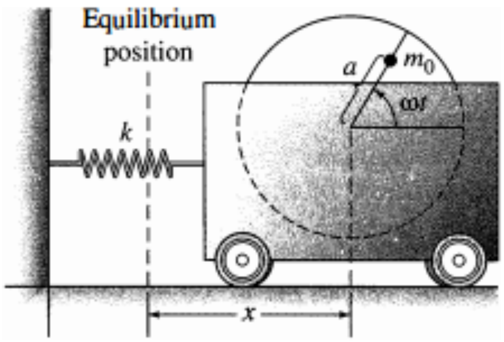
Solutions

3.6: Forced Oscillations and Resonance

In Section 3.4 we derived the differential equation

$$mx'' + cx' + kx = F(t). \tag{1}$$

We wish now to consider what happens when $F(t) = F_0 \cos \omega t$ or $F(t) = F_0 \sin \omega t$.



An example of when this can occur is when there is a rotating machine component involved in the mass which can provide a simple harmonic force. We arrive at the differential equation

$$mx'' + kx = F_0 \cos \omega t. \tag{2}$$

Undamped Forced Oscillations: To study the undamped oscillations under the influence of the external force $F(t) = F_0 \cos \omega t$, we set $c = 0$ in Equation (1) and begin with the equation

$$mx'' + kx = F_0 \cos \omega t \tag{3}$$

whose complimentary solution is $x_c = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$, where $\omega_0 = \sqrt{k/m}$ is the **natural frequency** of the mass-spring system. We can also see that the particular solution is of the form $x_p = A \cos \omega t$, where ω is the **circular frequency**. Suppose $\omega \neq \omega_0$. (Why?) Taking derivative of x_p and plugging into Equation (2), we get

$$-m\omega^2 \cos \omega t + kA \cos \omega t = F_0 \cos \omega t$$

so that

$$A = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2}. \tag{4}$$

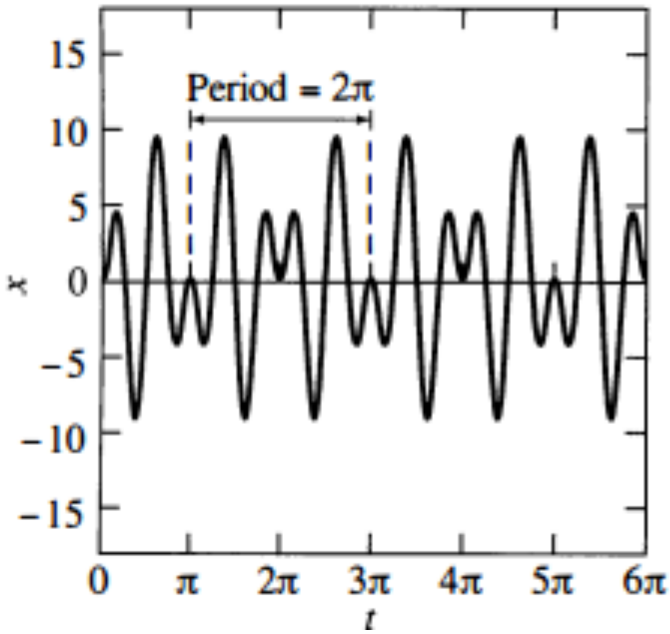
Therefore, the general solution $x = x_c + x_p$ is given by

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t. \tag{5}$$

Just as in Section 3.4, this becomes

$$x(t) = C \cos(\omega_0 t - \alpha) + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t. \tag{6}$$

Example 1. (Undamped Forced Oscillations)
Suppose that $m = 1$, $k = 9$, $F_0 = 80$ and $\omega = 5$ in (2). Find $x(t)$ if $x(0) = x'(0) = 0$.



$x'' + 9x = 80\cos 5t$

so $\omega_0 = \sqrt{k/m} = 3$ and

$x_c = c_1\cos 3t + c_2\sin 3t$.

Also, from Section 3.5, we know

$x_p = A\cos 5t + B\sin 5t$.

Thus

$x_p'' = -25A\cos 5t - 25B\sin 5t$

Solving, we find $A = -5$ and $B = 0$.

So $x_p = -5\cos 5t$.

Then $x = -5\cos 5t + c_1\cos 3t + c_2\sin 3t$

and $x' = 25\sin 5t - 3c_1\sin 3t + 3c_2\cos 3t$.

Plug in initial values to find

$c_1 = 5$ and $c_2 = 0$.

Finally,

$x = 5\cos 3t - 5\cos 5t$.

Beats:

If $x(0) = x'(0) = 0$ then the solution to (2) can be arranged as

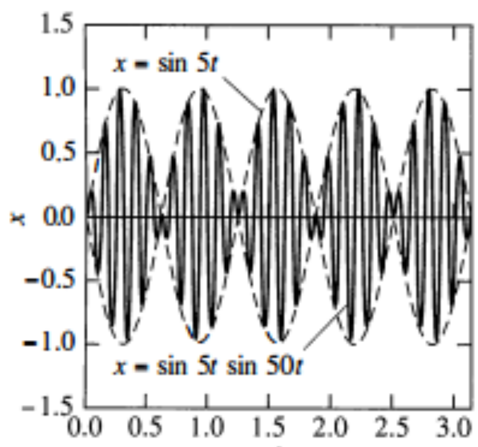
$$\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \tfrac{1}{2}(\omega_0 - \omega)t \sin \tfrac{1}{2}(\omega_0 + \omega)t.$$

We see that if $|\omega - \omega_0|$ is small we get a rapid oscillation plus a slowing varying amplitude.

Example 2. When $m = 0.1$, $F_0 = 50$, $\omega_0 = 55$ and $\omega = 45$ in (2), the solution written as above is given by

$$x(t) = \sin 5t \sin 50t$$

and the solution curve looks as below.



When $\omega_0 = \omega$ in (2) we see that the complementary and particular solutions would have the same form. In this case we see the phenomenon of resonance.

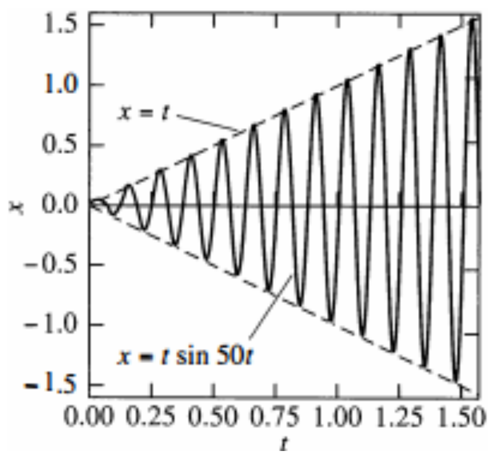
This phenomenon can be heard when two horns play simultaneously, but are not exactly attuned.

If $x(0) = x'(0) = 0$, then $c_1 = \frac{-F_0}{m(\omega_0^2 - \omega^2)}$ and $c_2 = 0$ in Equation (5).

So $x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos\omega t - \cos\omega_0 t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\frac{1}{2}(\omega_0 - \omega)t \sin\frac{1}{2}(\omega_0 + \omega)t.$

Note that if $\omega_0 \approx \omega$, then we have rapid oscillations with slowly varying amplitude $A(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\frac{1}{2}(\omega_0 - \omega)t.$

Example 3. (Resonance) Suppose that in (2) we have that $m = 5$ kg and $k = 500$ N/m. Then the natural frequency is $\omega_0 = 10$ rad/s. If the flywheel revolves at the same rate, then the solution curve looks as below.



Equation (3) becomes

$$5x'' + 500x = F_0 \cos 10t.$$

Then the natural frequency

$$w_0 = 10 \text{ rad/s} = w,$$

where w is the circular frequency of the flywheel in the first picture.

Therefore

$x_c = C_1 \cos 10t + C_2 \sin 10t$ and

$x_p = t(A\cos 10t + B \sin 10t)$

Recall the damping regimes for next section.

$$C_{cr} = \sqrt{4km}$$

Over damped: Two distinct real roots, $c > c_{cr}$

$$x_c = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Critically damped: Repeated real roots, $c = c_{cr}$

$$x_c = (c_1 + c_2 t) e^{rt}$$

Underdamped: Complex roots, $c < c_{cr}$

$$x_c = e^{-pt}(a \cos w_1 t + b \sin w_1 t)$$

More Complex Examples:

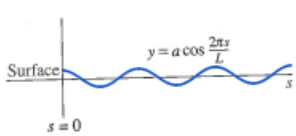


FIGURE 3.6.6. The washboard road surface of Example 5.

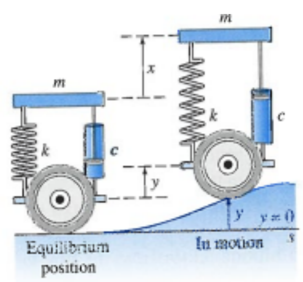


FIGURE 3.6.7. The "unicycle model" of a car.

Damped Forced Oscillations: Consider now the full generality of Equation (2):

$$cx' + mx'' + kx = F_0 \cos \omega t.$$

In this case, we can apply the same trig laws as in Section 3.4 to get

$$x_p = C \cos(\omega t - \alpha).$$

Example 4. Find the transient motion (x_c) and the steady periodic oscillations (x_p) of a damped mass-and-spring system with $m = 1$, $c = 2$, and $k = 26$ under the influence of an external force $F(t) = 82 \cos 4t$ with $x(0) = 6$ and $x'(0) = 0$. Also investigate the possibility of practical resonance for this system; i.e. what values of ω maximize the forced amplitude?

$x'' + 2x' + 26x = 82\cos 4t; \quad x(0) = 6, \quad x'(0) = 0$

$r^2 + 2r + 26 = (r+1)^2 + 25 = 0 \implies r = -1 \pm 5i$ and

$x_c = e^{-t}(c_1 \cos 5t + c_2 \sin 5t)$ (Underdamped)

$x_p = A\cos 4t + B \sin 4t. \implies$ Solve for A & B to get $x_p = 5\cos 4t + 4\sin 4t.$

Then $x(t) = e^{-t}(c_1 \cos 5t + c_2 \sin 5t) + 5\cos 4t + 4\sin 4t.$

and $x'(t) = -e^{-t}(c_1\cos 5t + c_2 \sin 5t) + e^{-t}(-5c_1 \sin 5t + 5c_2 \cos 5t) - 20\sin 4t + 16\cos 4t.$

Use initial values to get $c_1 = 1$ and $c_2 = -3$ so that

$$x(t) = e^{-t}(\cos 5t - 3\sin 5t) + 5\cos 4t + 4\sin 4t$$

$$= \underbrace{e^{-t}(\cos 5t - 3\sin 5t)}_{\text{transient motion}} + \underbrace{\sqrt{41}\,\cos(4t - \alpha)}_{\text{steady periodic oscillations.}}, \text{ where } \alpha = \tan^{-1}(4/5) \approx 0.6751$$

Suppose we wish to maximize the resonance (or possibly avoid the maximum), The practical resonance is where this maximum occurs. This depends on the circular frequency ω.

Consider $x'' + 2x' + 26x = 82\cos \omega t$. Then, if $\omega \neq \omega_0$, we have $x_p = A\cos\omega t + B\sin \omega t = C\cos(\omega t - \alpha)$, where $C = \sqrt{A^2 + B^2}$ is the amplitude of the steady periodic oscillation.

We can solve for A and B in general in Equation (2) to be $A = \dfrac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2}$ and $B = \dfrac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}.$

Thus

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

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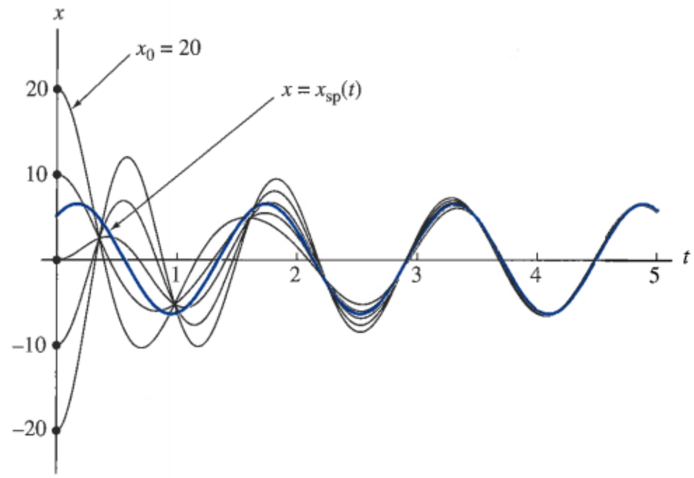

FIGURE 3.6.8. Solutions of the initial value problem in (24) with $x_0 = -20, -10, 0, 10,$ and 20.

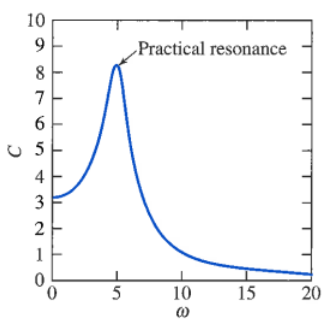

FIGURE 3.6.9. Plot of amplitude C versus external frequency ω.

To investigate the possibility of practical resonance in the given system, we substitute the values $m = 1$, $c = 2$, and $k = 26$ in (21) and find that the forced amplitude at frequency ω is

$$C(\omega) = \frac{82}{\sqrt{676 - 48\omega^2 + \omega^4}}.$$

The graph of $C(\omega)$ is shown in Fig. 3.6.9. The maximum amplitude occurs when

$$C'(\omega) = \frac{-41(4\omega^3 - 96\omega)}{(676 - 48\omega^2 + \omega^4)^{3/2}} = \frac{-164\omega(\omega^2 - 24)}{(676 - 48\omega^2 + \omega^4)^{3/2}} = 0.$$

Thus practical resonance occurs when the external frequency is $\omega = \sqrt{24}$ (a bit less than the mass-and-spring's undamped critical frequency of $\omega_0 = \sqrt{k/m} = \sqrt{26}$). ■

Homework. 1-6 (all)